

Evaluation of missing data mechanisms in two and three dimensional incomplete tables

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ABSTRACT. In this paper, we provide necessary conditions for the various missing mechanisms of a variable in terms of response and nonresponse odds for two and three dimensional incomplete tables. Log-linear parametrization and some properties of the missing data models for the above tables are discussed. All possible cases in which data on one, two or all variables may be missing are considered. For sensitivity analysis of the incomplete tables, we suggest some easily verifiable procedures to evaluate the missing at random (MAR), missing completely at random (MCAR) and not missing at random (NMAR) assumptions in the missing data models. These methods depend only on joint and marginal odds computed from fully and partially observed counts in the tables, respectively. Finally, some real-life datasets are analyzed to illustrate our results.

1. INTRODUCTION

A contingency table with fully observed counts and supplemental margins (nonresponses) is called an incomplete table. For inference purposes, three types of missing data mechanisms are used to study nonresponses (see Little and Rubin (2002)) : missing completely at random (MCAR), missing at random (MAR) and not missing at random (NMAR). The missing mechanism is said to be MCAR when missingness is independent of both observed and unobserved data, MAR when missingness depends only on observed data, and NMAR if missingness depends only on unobserved data. Nonresponses can be either ignorable (when the missing data mechanism is MAR or MCAR, and the estimated parameters are distinct from those involving the missing data mechanism) or nonignorable (when the missing data mechanism is NMAR).

The assumption regarding the missing data mechanism in the model cannot be usually confirmed from the model fit to the observed data. Hence, it is difficult to use nonresponse models to analyze incomplete tables. Molenberghs *et al.* (2008) demonstrated that there is a MAR model for every NMAR model, which gives the same fit to the observed data. Several researchers have implemented sensitivity analysis to assess the missing data mechanism in incomplete tables. One approach is to compare the relevant parameter estimates from a range of candidate models (see Baker *et al.* (1992)). Another approach is to consider overspecified models with sensitivity parameter and construct confidence intervals for the parameters to investigate the statistical uncertainty due to incomplete data and finite sampling (see Molenberghs *et al.* (2001); Vansteelandt *et al.* (2006)).

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Recently, Kim *et al.* (2015) proposed a new, convenient method of sensitivity analysis to assess the MAR assumption for a two-way incomplete table with one supplemental margin ($I \times J \times 2$ table in Baker and Laird (1988)). In this paper, we advance such methods for $I \times J \times 2 \times 2$, $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ tables, that is, two-way tables with both variables missing, and three way tables with one, two and all variables missing, respectively. Note that the evaluation of missing mechanisms in the above incomplete tables has not been studied earlier in the literature.

The remaining part of the paper is organized as follows. Section 2 considers nonresponse log-linear models for $I \times J \times 2 \times 2$ tables. Some results regarding characteristic features of these missing data models are provided. The missing mechanisms of the variables are identified using the response and nonresponse odds based on cell probabilities. Assessment of MAR, MCAR and NMAR mechanisms is carried out using the estimators of the above odds computed from only the observed counts in the tables. Therefore, there is no need to use numerical or simulation procedures for the analysis of such tables. The results and discussions in Section 2 are extended in Section 3 to three-way incomplete tables with one, two or all variables missing. Section 4 presents real-life data analysis to illustrate the results in Sections 2 and 3. Some concluding remarks about the methods of assessment for the missing mechanisms are provided in Section 5.

2. MISSING DATA MODELS FOR THE $I \times J \times 2 \times 2$ TABLE

Kim *et al.* (2015) considered missing data models and sensitivity analysis for the $I \times J \times 2$ table. They also mentioned that it would be of interest to study such models and develop sensitivity analysis for two way tables with both variables subject to missingness. In this section, we address these issues.

Let Y_1 and Y_2 be two categorical variables with I and J levels respectively. It is assumed that data on both variables may be missing. For $i = 1, 2$, let R_i denote the missing indicator variable for Y_i such that $R_i = 1$ if Y_i is observed and $R_i = 2$ otherwise. Then we have an $I \times J \times 2 \times 2$ table with cell probabilities $\pi = \{\pi_{ijkl}\}$ and cell counts $\mathbf{y} = \{y_{ijkl}\}$, where $1 \leq i \leq I$, $1 \leq j \leq J$ and $k, l = 1, 2$. The vector of observed frequencies is given by $\mathbf{y}_{\text{obs}} = (\{y_{ij11}\}, \{y_{+j21}\}, \{y_{i+12}\}, y_{++22})$, where a '+' in the subscript denotes summation over levels of the corresponding variable. Here $\{y_{ij11}\}$ are the fully observed counts, while $\{y_{+j21}\}$, $\{y_{i+12}\}$ and y_{++22} are the supplemental margins. Table 1 below shows the $I \times J \times 2 \times 2$ table.

Table 1. $I \times J \times 2 \times 2$ Incomplete Table.

		$R_2 = 1$				$R_2 = 2$
		$Y_2 = 1$	$Y_2 = 2$	\cdots	$Y_2 = J$	Y_2 missing
$R_1 = 1$	$Y_1 = 1$	y_{1111}	y_{1211}	\cdots	y_{1J11}	y_{1+12}
	$Y_1 = 2$	y_{2111}	y_{2211}	\cdots	y_{2J11}	y_{2+12}
	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
	$Y_1 = I$	y_{I111}	y_{I211}	\cdots	y_{IJ11}	y_{I+12}
$R_1 = 2$	Y_1 missing	y_{+121}	y_{+221}	\cdots	y_{+J21}	y_{++22}

Let the vector of expected counts be $\mu = \{\mu_{ijkl}\}$ and $N = \sum_{i,j,k,l} y_{ijkl}$ be the total cell count. Under Poisson sampling for observed cell counts, the log-likelihood of μ is

$$(2.1) \quad \begin{aligned} l(\mu; \mathbf{y}_{\text{obs}}) = & \sum_{i,j} y_{ij11} \log \mu_{ij11} + \sum_j y_{+j21} \log \mu_{+j21} + \sum_i y_{i+12} \log \mu_{i+12} \\ & + y_{++22} \log \mu_{++22} - \sum_{i,j,k,l} \mu_{ijkl} + \Delta, \end{aligned}$$

where Δ is some constant. For an $I \times J \times 2 \times 2$ incomplete table, Baker *et al.* (1992) proposed the following log-linear model (with no three-way or four-way interactions):

$$(2.2) \quad \begin{aligned} \log \mu_{ijkl} = & \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) \\ & + \lambda_{Y_1 R_1}(i, k) + \lambda_{Y_2 R_1}(j, k) + \lambda_{Y_1 R_2}(i, l) + \lambda_{Y_2 R_2}(j, l) + \lambda_{R_1 R_2}(k, l). \end{aligned}$$

Each log-linear parameter in (2.2) satisfies the constraint that the sum over each of its arguments is 0. Henceforth, in this paper, we use the missingness per variable concept and not the missingness of the outcome vector as a whole. This is based on the classification scheme of missing data models considered in Park *et al.* (2014). Using this scheme and (2.2), Baker *et al.* (1992) suggested nine identifiable missing data models, whose log-linear formulations (based on different missing mechanisms for Y_1 and Y_2) are as follows (see Park *et al.* (2014)).

M1. NMAR for Y_1 , MCAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 R_1}(i, k) + \lambda_{R_1 R_2}(k, l)$$

M2. NMAR for Y_1 , MAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 R_1}(i, k) + \lambda_{Y_1 R_2}(i, l) + \lambda_{R_1 R_2}(k, l)$$

M3. NMAR for both Y_1 and Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 R_1}(i, k) + \lambda_{Y_2 R_2}(j, l) + \lambda_{R_1 R_2}(k, l)$$

M4. MAR for Y_1 , MCAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_2 R_1}(j, k) + \lambda_{R_1 R_2}(k, l)$$

M5. MAR for both Y_1 and Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_2 R_1}(j, k) + \lambda_{Y_1 R_2}(i, l) + \lambda_{R_1 R_2}(k, l)$$

M6. MAR for Y_1 , NMAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_2 R_1}(j, k) + \lambda_{Y_2 R_2}(j, l) + \lambda_{R_1 R_2}(k, l)$$

M7. MCAR for Y_1 , MAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 R_2}(i, l) + \lambda_{R_1 R_2}(k, l)$$

M8. MCAR for Y_1 , NMAR for Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_2 R_2}(j, l) + \lambda_{R_1 R_2}(k, l)$$

M9. MCAR for both Y_1 and Y_2 :

$$\log \mu_{ijkl} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{R_1}(k) + \lambda_{R_2}(l) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{R_1 R_2}(k, l)$$

Next, we describe various features of these models, which help us investigate the missing data mechanisms of the variables in an $I \times J \times 2 \times 2$ table.

2.1. Properties of the missing data models. Following Park *et al.* (2014), define the following odds for a pair (j, j') of Y_2 :

$$\nu_i(j, j') = \frac{\pi_{ij11}}{\pi_{ij'11}}, \quad \nu_n(j, j') = \min_i \{\nu_i(j, j')\}, \quad \nu_m(j, j') = \max_i \{\nu_i(j, j')\}, \quad \nu(j, j') = \frac{\pi_{+j21}}{\pi_{+j'21}},$$

where $\nu_i(j, j')$ are the response odds and $\nu(j, j')$ are the nonresponse odds when Y_1 is missing. Similarly, define the following odds for any pair (i, i') of Y_1 :

$$\omega_j(i, i') = \frac{\pi_{ij11}}{\pi_{i'j11}}, \quad \omega_n(i, i') = \min_j \{\omega_j(i, i')\}, \quad \omega_m(i, i') = \max_j \{\omega_j(i, i')\}, \quad \omega(i, i') = \frac{\pi_{i+12}}{\pi_{i'+12}}.$$

Here, $\omega_j(i, i')$ are the response odds, while $\omega(i, i')$ are the nonresponse odds when Y_2 is missing. Let $OI(i, i') = (\omega_n(i, i'), \omega_m(i, i'))$ and $OI(j, j') = (\nu_n(j, j'), \nu_m(j, j'))$. Note that the interval $OI(i, i')$ contains $\omega_j(i, i')$, while the interval $OI(j, j')$ contains $\nu_i(j, j')$. We now study the behaviour of the above odds under some of the nine models.

1. Model M3 (NMAR for both Y_1 and Y_2) :

Under Model M3, it can be shown that for any pair (i, i') of Y_1 , we have

$$\begin{aligned} \omega_j(i, i') &= \exp\{\lambda_{Y_1}(i) - \lambda_{Y_1}(i') + \lambda_{Y_1Y_2}(i, j) - \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1R_1}(i, 1) - \lambda_{Y_1R_1}(i', 1)\}, \\ \omega(i, i') &= \frac{\sum_j \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1R_1}(i, 1) + \lambda_{Y_2R_2}(j, 2)\}}{\sum_j \exp\{\lambda_{Y_1}(i') + \lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1R_1}(i', 1) + \lambda_{Y_2R_2}(j, 2)\}}, \\ \frac{\omega_m(i, i')}{\omega(i, i')} &= \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1Y_2}(i, m) + \lambda_{Y_2R_2}(j, 2)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(i', m) + \lambda_{Y_2R_2}(j, 2)\}}, \\ \frac{\omega_n(i, i')}{\omega(i, i')} &= \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1Y_2}(i, n) + \lambda_{Y_2R_2}(j, 2)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(i', n) + \lambda_{Y_2R_2}(j, 2)\}}. \end{aligned}$$

Now

$$\begin{aligned} &\omega_j(i, i') < \omega_m(i, i') \\ \Rightarrow &1 < \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1Y_2}(i, m) + \lambda_{Y_2R_2}(j, 2)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(i', m) + \lambda_{Y_2R_2}(j, 2)\}} \\ \Rightarrow &\frac{\omega_m(i, i')}{\omega(i, i')} > 1. \end{aligned}$$

Also,

$$\begin{aligned} &\omega_j(i, i') > \omega_n(i, i') \\ \Rightarrow &1 > \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1Y_2}(i, n) + \lambda_{Y_2R_2}(j, 2)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(i', n) + \lambda_{Y_2R_2}(j, 2)\}} \\ \Rightarrow &\frac{\omega_n(i, i')}{\omega(i, i')} < 1. \end{aligned}$$

Hence, $\omega(i, i') \in (\omega_n(i, i'), \omega_m(i, i')) = OI(i, i')$ if $|\lambda_{Y_2R_2}(j, 2)| < \infty$. Now consider the response and non-response odds based on π for any pair (j, j') of Y_2 . Then using similar arguments as above, it can be shown that $\nu(j, j') \in (\nu_n(j, j'), \nu_m(j, j')) = OI(j, j')$ if $|\lambda_{Y_1R_1}(i, 2)| < \infty$. Thus under Model M3, both the following conditions hold :

- (i) $\omega(i, i') \in OI(i, i')$ for any pair (i, i') of Y_1 if $|\lambda_{Y_2 R_2}(j, 2)| < \infty$,
- (ii) $\nu(j, j') \in OI(j, j')$ for any pair (j, j') of Y_2 if $|\lambda_{Y_1 R_1}(i, 2)| < \infty$.

2. Model M5 (MAR for both Y_1 and Y_2) :

Under Model M5, it can be shown that for any pair (i, i') of Y_1 , we have

$$\begin{aligned}
\omega_j(i, i') &= \exp\{\lambda_{Y_1}(i) - \lambda_{Y_1}(i') + \lambda_{Y_1 Y_2}(i, j) - \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 R_2}(i, 1) \\
&\quad - \lambda_{Y_1 R_2}(i', 1)\}, \\
\omega(i, i') &= \frac{\sum_j \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 R_2}(i, 2) + \lambda_{Y_2 R_1}(j, 1)\}}{\sum_j \exp\{\lambda_{Y_1}(i') + \lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 R_2}(i', 2) + \lambda_{Y_2 R_1}(j, 1)\}}, \\
\frac{\omega_m(i, i')}{\omega(i, i')} &= \exp\{2(\lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2))\} \times B_m(i, i'), \\
\frac{\omega_n(i, i')}{\omega(i, i')} &= \exp\{2(\lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2))\} \times B_n(i, i'),
\end{aligned}$$

where

$$\begin{aligned}
B_m(i, i') &= \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 Y_2}(i, m) + \lambda_{Y_2 R_1}(j, 1)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(i', m) + \lambda_{Y_2 R_1}(j, 1)\}}, \\
B_n(i, i') &= \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 Y_2}(i, n) + \lambda_{Y_2 R_1}(j, 1)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(i', n) + \lambda_{Y_2 R_1}(j, 1)\}}.
\end{aligned}$$

Now

$$\begin{aligned}
&\omega_j(i, i') < \omega_m(i, i') \\
\Rightarrow 1 &< \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 Y_2}(i, m) + \lambda_{Y_2 R_1}(j, 1)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(i', m) + \lambda_{Y_2 R_1}(j, 1)\}} \\
\Rightarrow B_m(i, i') &> 1.
\end{aligned}$$

Also,

$$\begin{aligned}
&\omega_j(i, i') > \omega_n(i, i') \\
\Rightarrow 1 &> \frac{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i', j) + \lambda_{Y_1 Y_2}(i, n) + \lambda_{Y_2 R_1}(j, 1)\}}{\sum_j \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(i', n) + \lambda_{Y_2 R_1}(j, 1)\}} \\
\Rightarrow B_n(i, i') &< 1.
\end{aligned}$$

Now consider the response and non-response odds based on π for any pair (j, j') of Y_2 . Then using similar arguments as above, it can be shown that

$$\begin{aligned}
\frac{\nu_m(j, j')}{\nu(j, j')} &= \exp\{2(\lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2))\} \times A_m(j, j'), \\
\frac{\nu_n(j, j')}{\nu(j, j')} &= \exp\{2(\lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2))\} \times A_n(j, j'),
\end{aligned}$$

where

$$A_m(j, j') = \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(m, j) + \lambda_{Y_1 R_2}(i, 1)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(m, j') + \lambda_{Y_1 R_2}(i, 1)\}},$$

$$A_n(j, j') = \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(n, j) + \lambda_{Y_1 R_2}(i, 1)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(n, j') + \lambda_{Y_1 R_2}(i, 1)\}}.$$

Now $\nu_i(j, j') < \nu_m(j, j') \Rightarrow A_m(j, j') > 1$ and $\nu_i(j, j') > \nu_n(j, j') \Rightarrow A_n(j, j') < 1$. Suppose $\omega(i, i') \in OI(i, i') \Leftrightarrow \frac{\omega_m(i, i')}{\omega(i, i')} > 1$ and $\frac{\omega_n(i, i')}{\omega(i, i')} < 1$. Then

$$\frac{\omega_m(i, i')}{\omega(i, i')} > 1 \Leftrightarrow \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) > -\frac{1}{2} \log B_m(i, i').$$

Also,

$$\frac{\omega_n(i, i')}{\omega(i, i')} < 1 \Leftrightarrow \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_n(i, i').$$

Hence $\omega(i, i') \in OI(i, i')$ iff $-\frac{1}{2} \log B_m(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_n(i, i')$. Again suppose $\nu(j, j') \in OI(j, j') \Leftrightarrow \frac{\nu_m(j, j')}{\nu(j, j')} > 1$ and $\frac{\nu_n(j, j')}{\nu(j, j')} < 1$. Then using similar arguments as above, we have $\nu(j, j') \in OI(j, j')$ iff $-\frac{1}{2} \log A_m(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A_n(j, j')$. Thus under Model M5, both the following conditions hold :

- (i) $\omega(i, i') \in OI(i, i')$ iff $-\frac{1}{2} \log B_m(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_n(i, i')$,
- (ii) $\nu(j, j') \in OI(j, j')$ iff $-\frac{1}{2} \log A_m(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A_n(j, j')$.

Similar conditions can be obtained under the other models. Let $A'_m(j, j')$ and $A'_n(j, j')$ denote $A_m(j, j')$ and $A_n(j, j')$ respectively with $\lambda_{Y_1 R_2}(i, 2) = 0$. Also, let $B'_m(i, i')$ and $B'_n(i, i')$ denote $B_m(i, i')$ and $B_n(i, i')$ respectively with $\lambda_{Y_2 R_1}(j, 2) = 0$. Then Table 2 summarizes the conditions under which $\omega(i, i') \in OI(i, i')$ for any pair (i, i') of Y_1 and $\nu(j, j') \in OI(j, j')$ for any pair (j, j') of Y_2 for Models M1-M9.

Table 2. Conditions under missing data models for an $I \times J \times 2 \times 2$ Incomplete Table.

Model	Conditions
Model M1	$ \lambda_{Y_1 R_1}(i, 2) < \infty$
Model M2	$-\frac{1}{2} \log B'_m(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B'_n(i, i'); \lambda_{Y_1 R_1}(i, 2) < \infty$
Model M3	$ \lambda_{Y_1 R_1}(i, 2) < \infty; \lambda_{Y_2 R_2}(j, 2) < \infty$
Model M4	$-\frac{1}{2} \log A'_m(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A'_n(j, j')$
Model M5	$-\frac{1}{2} \log B_m(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_n(i, i');$ $-\frac{1}{2} \log A_m(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A_n(j, j')$
Model M6	$-\frac{1}{2} \log A'_m(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A'_n(j, j'); \lambda_{Y_2 R_2}(j, 2) < \infty$
Model M7	$-\frac{1}{2} \log B'_m(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B'_n(i, i')$
Model M8	$ \lambda_{Y_2 R_2}(j, 2) < \infty$
Model M9	Nil

Denote $A_m^*(j, j')$ to be $A_m(j, j')$ under Model M5 and $A'_m(j, j')$ under Models M4 and M6. Similarly, define $A_n^*(j, j')$. Now denote $B_m^*(i, i')$ to be $B_m(i, i')$ under Model M5 and $B'_m(i, i')$ under Models M2 and M7. Similarly, define $B_n^*(i, i')$. Then using properties of the nine

models, the next two theorems give necessary conditions for the missing data mechanisms of variables in an $I \times J \times 2 \times 2$ table.

Theorem 2.1. Under Models M1-M9 for an $I \times J \times 2 \times 2$ table, we have the following cases corresponding to the missing mechanism of Y_1 .

- (a) If Y_1 has a MCAR or NMAR mechanism, then $\nu(j, j') \in OI(j, j')$ if $|\lambda_{Y_1 R_1}(i, 2)| < \infty$.
- (b) If Y_1 has a MAR mechanism, then only one of the following conditions holds for each pair (j, j') of Y_2 :
 - (i) $\nu(j, j') \in OI(j, j')$ iff $-\frac{1}{2} \log A_m^*(j, j') < \lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A_n^*(j, j')$,
 - (ii) $\nu(j, j') \notin OI(j, j')$ iff $\lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) > -\frac{1}{2} \log A_n^*(j, j')$ or $\lambda_{Y_2 R_1}(j', 2) - \lambda_{Y_2 R_1}(j, 2) < -\frac{1}{2} \log A_m^*(j, j')$.

Theorem 2.2. Under Models M1-M9 for an $I \times J \times 2 \times 2$ table, we have the following cases corresponding to the missing mechanism of Y_2 .

- (a) If Y_2 has a MCAR or NMAR mechanism, then $\omega(i, i') \in OI(i, i')$ if $|\lambda_{Y_2 R_2}(j, 2)| < \infty$.
- (b) If Y_2 has a MAR mechanism, then only one of the following conditions holds for each pair (i, i') of Y_1 :
 - (i) $\omega(i, i') \in OI(i, i')$ iff $-\frac{1}{2} \log B_m^*(i, i') < \lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_n^*(i, i')$,
 - (ii) $\omega(i, i') \notin OI(i, i')$ iff $\lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) > -\frac{1}{2} \log B_n^*(i, i')$ or $\lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2) < -\frac{1}{2} \log B_m^*(i, i')$.

Remark 2.1. Note that if $\lambda_{Y_1 R_2}(i, l) = 0 \forall i, l$, then the MAR mechanism of Y_2 reduces to the MCAR mechanism. Also, $\omega_m(i, i')/\omega(i, i') = B_m(i, i') > 1$ and $\omega_n(i, i')/\omega(i, i') = B_n(i, i') < 1$ imply $\omega(i, i') \in OI(i, i')$, which is exactly one of the conditions under MCAR models for Y_2 (Models M1, M4 and M9). Further note that $B_m(j, j')$ and $B_n(j, j')$ are independent of $\lambda_{Y_1}(i)$'s, $\lambda_{R_1}(k)$'s and $\lambda_{R_2}(l)$'s. Similar results hold for the MAR mechanism of Y_1 if $\lambda_{Y_2 R_1}(j, k) = 0 \forall j, k$.

Remark 2.2. From Theorems 2.1 and 2.2, note that if the missing mechanism of Y_1 or Y_2 is NMAR or MCAR, then $\nu(j, j') \in OI(j, j')$ for any pair (j, j') of Y_2 and $\omega(i, i') \in OI(i, i')$ for any pair (i, i') of Y_1 . Also, if there exists at least one pair (i, i') of Y_1 such that $\omega(i, i') \notin OI(i, i')$, then $|\lambda_{Y_1 R_2}(i', 2) - \lambda_{Y_1 R_2}(i, 2)|$ is larger than that when $\omega(i, i') \in OI(i, i')$. We say that the missing mechanism of Y_2 is strong MAR in the first case and non-strong (weak) in the second one. Similar results hold for the MAR mechanism of Y_1 .

Remark 2.3. If $I = J = 2$, then we have a $2 \times 2 \times 2 \times 2$ table. Under Models M1-M9,

$$\lambda_{Y_1 Y_2}(1, 1) = \frac{1}{4} \log \left[\frac{\nu_1(1, 2)}{\nu_2(1, 2)} \right] = \frac{1}{4} \log \left[\frac{\omega_1(1, 2)}{\omega_2(1, 2)} \right].$$

Hence, for fixed π , the length of $OI(1, 2) = |\nu_1(1, 2) - \nu_2(1, 2)|$ and that of $OI'(1, 2) = |\omega_1(1, 2) - \omega_2(1, 2)|$ or equivalently the sizes of the parameter regions for the weak MAR mechanisms of Y_1 and Y_2 respectively are each directly proportional to $|\lambda_{Y_1 Y_2}(1, 1)|$, the strength of association between Y_1 and Y_2 .

2.2. Assessment of the MCAR, NMAR and MAR mechanisms. A perfect fit model is one in which the estimated expected counts are equal to the observed counts. It is known that Models M1, M4, M7, M8 and M9 do not provide perfect fits for observed counts in the

tables (see Table II on p. 647 of Baker *et al.* (1992)). However, Models M2, M3, M5 and M6 are perfect fit models so that $\hat{\pi}_{ij11} = y_{ij11}/N$, $\hat{\pi}_{i+12} = y_{i+12}/N$ and $\hat{\pi}_{+j21} = y_{+j21}/N$ (see Table II on p. 648 of Baker *et al.* (1992)). Hence, the estimators of the various odds under them are as follows.

$$\begin{aligned}\hat{\nu}_i(j, j') &= \frac{y_{ij11}}{y_{ij'11}}, \quad \hat{\nu}_n(j, j') = \min_i \{\hat{\nu}_i(j, j')\}, \quad \hat{\nu}_m(j, j') = \max_i \{\hat{\nu}_i(j, j')\}, \quad \hat{\nu}(j, j') = \frac{y_{+j21}}{y_{+j'21}}; \\ \hat{\omega}_j(i, i') &= \frac{y_{ij11}}{y_{i'j11}}, \quad \hat{\omega}_n(i, i') = \min_j \{\hat{\omega}_j(i, i')\}, \quad \hat{\omega}_m(i, i') = \max_j \{\hat{\omega}_j(i, i')\}, \quad \hat{\omega}(i, i') = \frac{y_{i+12}}{y_{i'+12}}.\end{aligned}$$

Note that the estimated expected counts under non-perfect fit models are non-trivial functions of the observed counts in the tables. This implies that the MLE's of the response and the nonresponse odds under such models are more difficult to calculate than those under perfect fit models. For example, the estimated cell probabilities under Model M1 (see p. 647 in Baker *et al.* (1992)) are

$$\hat{\pi}_{ij11} = \frac{y_{ij11}y_{i+1+}y_{++11}}{Ny_{i+11}y_{++1+}}, \quad \hat{\pi}_{+j21} = \frac{y_{+j21}}{N}, \quad \hat{\pi}_{i+12} = \frac{y_{++12} \sum_j \hat{\pi}_{ij11}}{y_{++11}} = \frac{y_{i+1+}y_{++12}}{Ny_{++1+}}.$$

Hence, the MLE's of the response and nonresponse odds under Model M1 are

$$\begin{aligned}\hat{\nu}_i(j, j') &= \frac{y_{ij11}}{y_{ij'11}}, \quad \hat{\nu}_n(j, j') = \min_i \{\hat{\nu}_i(j, j')\}, \quad \hat{\nu}_m(j, j') = \max_i \{\hat{\nu}_i(j, j')\}, \quad \hat{\nu}(j, j') = \frac{y_{+j21}}{y_{+j'21}}; \\ \hat{\omega}_j(i, i') &= \frac{y_{ij11}y_{i+1+}y_{i'+11}}{y_{i'j11}y_{i+11}y_{i'+1+}}, \quad \hat{\omega}_n(i, i') = \min_j \{\hat{\omega}_j(i, i')\}, \quad \hat{\omega}_m(i, i') = \max_j \{\hat{\omega}_j(i, i')\}, \\ \hat{\omega}(i, i') &= \frac{\sum_j \hat{\pi}_{ij11}}{\sum_j \hat{\pi}_{i'j11}} = \frac{y_{i+1+}}{y_{i'+1+}}.\end{aligned}$$

Similarly, the MLE's of the response and nonresponse odds under Model M8 (MCAR for Y_1 , NMAR for Y_2) may be obtained. Let $\widehat{OI}(i, i') = (\hat{\omega}_n(i, i'), \hat{\omega}_m(i, i'))$ and $\widehat{OI}(j, j') = (\hat{\nu}_n(j, j'), \hat{\nu}_m(j, j'))$. Then the corollary below follows from Theorems 2.1 and 2.2, and Remark 2.2.

Corollary 2.1. For an $I \times J \times 2 \times 2$ table, if $\hat{\omega}(i, i') \notin \widehat{OI}(i, i')$ for at least one pair (i, i') of Y_1 or $\hat{\nu}(j, j') \notin \widehat{OI}(j, j')$ for at least one pair (j, j') of Y_2 , then the missing data mechanism of Y_1 or Y_2 is more likely to be MAR, but neither NMAR nor MCAR.

Remark 2.4. Note that only observed counts or their functions are used in Corollary 2.1 to assess the MCAR, MAR and NMAR mechanisms of the variables in an $I \times J \times 2 \times 2$ table. Since Corollary 2.1 follows from Theorems 2.1(a) and 2.2(a), it suffices to consider estimators of response and nonresponse odds under models in which the missing mechanism of Y_1 or Y_2 is MCAR or NMAR (Models M1, M3, M8 and M9) for the purpose of assessment. Also, our results in this section along with those obtained by Kim *et al.* (2015) completely establish the necessary conditions for the missing mechanisms of variables in two-way incomplete tables.

3. MISSING DATA MODELS FOR THREE-WAY INCOMPLETE TABLES

In this section, we propose log-linear models for three-way incomplete tables and study missing data mechanisms of the variables using these models. We also develop sensitivity

analysis for such tables. Suppose Y_1 , Y_2 and Y_3 are three categorical variables with I , J and K levels respectively. Then we have the following cases.

3.1. Case 1: One of the variables is missing. Without loss of generality (WLOG), let Y_1 be missing and R denote the missing indicator for Y_1 such that $R = 1$ if Y_1 is observed and $R = 2$ otherwise. Then for Y_1 , Y_2 , Y_3 and R , we have an $I \times J \times K \times 2$ table with cell counts $\mathbf{y} = \{y_{ijk r}\}$, where $1 \leq i \leq I$, $1 \leq j \leq J$, $1 \leq k \leq K$ and $r = 1, 2$. The vector of observed counts is $\mathbf{y}_{\text{obs}} = (\{y_{ijk1}\}, \{y_{+jk2}\})$, where $\{y_{ijk1}\}$ are the fully observed counts and $\{y_{+jk2}\}$ are the supplementary margins with ‘+’ representing summation over levels of the corresponding variable. For $I = J = K = 2$, the $2 \times 2 \times 2 \times 2$ incomplete table is given below.

Table 3. $2 \times 2 \times 2 \times 2$ Incomplete Table.

			$Y_3 = 1$	$Y_3 = 2$
$R = 1$	$Y_1 = 1$	$Y_2 = 1$	y_{1111}	y_{1121}
		$Y_2 = 2$	y_{1211}	y_{1221}
	$Y_1 = 2$	$Y_2 = 1$	y_{2111}	y_{2121}
		$Y_2 = 2$	y_{2211}	y_{2221}
$R = 2$	Missing	$Y_2 = 1$	y_{+112}	y_{+122}
		$Y_2 = 2$	y_{+212}	y_{+222}

Let $\pi = \{\pi_{ijk r}\}$ be the vector of cell probabilities, $\mu = \{\mu_{ijk r}\}$ be the vector of expected counts and $N = \sum_{i,j,k,r} y_{ijk r}$ be the total cell count. Under Poisson sampling for observed cell counts, the log-likelihood kernel of μ is

$$(3.1) \quad l(\mu; \mathbf{y}_{\text{obs}}) = \sum_{i,j,k} y_{ijk1} \log \mu_{ijk1} + \sum_{j,k} y_{+jk2} \log \mu_{+jk2} - \sum_{i,j,k,r} \mu_{ijk r}.$$

The log-linear model (with no three-way or four-way interactions) is then

$$(3.2) \quad \begin{aligned} \log \mu_{ijk r} = & \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(r) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) \\ & + \lambda_{Y_1 R}(i, r) + \lambda_{Y_2 R}(j, r) + \lambda_{Y_3 R}(k, r). \end{aligned}$$

We avoid higher order interactions in (3.2) since they are difficult to interpret and maximum likelihood estimation of the parameters becomes complicated. Each log-linear parameter in (3.2) satisfies the constraint that the sum over each of its arguments is 0. It is assumed in this case and subsequent ones that the missing mechanism of a variable may depend on itself (NMAR) or on one of the other variables (MAR) or none (MCAR). Accordingly, the various missing data models, which are submodels of (3.2), are as follows.

C1. NMAR for Y_1 :

$$\log \mu_{ijk r} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(r) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_1 R}(i, r)$$

C2. MAR for Y_1 (missing mechanism depends on Y_2) :

$$\log \mu_{ijk r} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(r) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_2 R}(j, r)$$

C3. MAR for Y_1 (missing mechanism depends on Y_3) :

$$\log \mu_{ijk r} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(r) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_3 R}(k, r)$$

C4. MCAR for Y_1 :

$$\log \mu_{ijk r} = \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_R(r) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k)$$

Each log-linear parameter in the above models satisfies the constraint that the sum over each of its arguments is 0. Also, for each model, there is an association term between a variable and its missing indicator if the missing mechanism is NMAR for that variable (for example, the term $\lambda_{Y_1 R}(i, r)$ in Model C1), between a variable and some other missing indicator if the missing mechanism is MAR for that variable (for example, the term $\lambda_{Y_2 R}(j, r)$ in Model C2) and none if the missing mechanism is MCAR for a variable (for example, $\lambda_{Y_1 R}(i, r)$, $\lambda_{Y_2 R}(j, r)$ and $\lambda_{Y_3 R}(k, r)$ are absent in Model C4).

3.1.1. *Properties of the missing data models.* Define the following odds for any pair (j, j') of Y_2 and $1 \leq k \leq K$:

$$\nu_{ik}(j, j') = \frac{\pi_{ijk1}}{\pi_{ij'k1}}, \quad \nu_{nk}(j, j') = \min_i \{\nu_{ik}(j, j')\}, \quad \nu_{mk}(j, j') = \max_i \{\nu_{ik}(j, j')\}, \quad \nu_k(j, j') = \frac{\pi_{+jk2}}{\pi_{+j'k2}}.$$

Similarly, define the following odds for any pair (k, k') of Y_3 and $1 \leq j \leq J$:

$$\nu_{ij}(k, k') = \frac{\pi_{ijk1}}{\pi_{ijk'1}}, \quad \nu_{nj}(k, k') = \min_i \{\nu_{ij}(k, k')\}, \quad \nu_{mj}(k, k') = \max_i \{\nu_{ij}(k, k')\}, \quad \nu_j(k, k') = \frac{\pi_{+jk2}}{\pi_{+jk'2}}.$$

Let $OI_k(j, j') = (\nu_{nk}(j, j'), \nu_{mk}(j, j'))$ and $OI_j(k, k') = (\nu_{nj}(k, k'), \nu_{mj}(k, k'))$. We next study the behaviour of the above odds under two of the four models.

1. Model C1 (NMAR for Y_1) :

Under Model C1, we have

$$\begin{aligned} \nu_{ik}(j, j') &= \exp\{\lambda_{Y_2}(j) - \lambda_{Y_2}(j') + \lambda_{Y_1 Y_2}(i, j) - \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_2 Y_3}(j, k) \\ &\quad - \lambda_{Y_2 Y_3}(j', k)\}, \\ \nu_k(j, j') &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_1 R}(i, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j') + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j', k) + \lambda_{Y_1 R}(i, 2)\}}, \\ \frac{\nu_{mk}(j, j')}{\nu_k(j, j')} &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(m, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(m, j') + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}, \\ \frac{\nu_{nk}(j, j')}{\nu_k(j, j')} &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(n, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(n, j') + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}. \end{aligned}$$

Now

$$\begin{aligned} &\nu_{ik}(j, j') < \nu_{mk}(j, j') \\ \Rightarrow 1 &< \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(m, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(m, j') + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}} \\ \Rightarrow \nu_{mk}(j, j') &> \nu_k(j, j'). \end{aligned}$$

Also,

$$\begin{aligned} &\nu_{ik}(j, j') > \nu_{nk}(j, j') \\ \Rightarrow 1 &> \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j') + \lambda_{Y_1 Y_2}(n, j) + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1 Y_2}(i, j) + \lambda_{Y_1 Y_2}(n, j') + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_1 R}(i, 2)\}} \\ \Rightarrow \nu_{nk}(j, j') &< \nu_k(j, j'). \end{aligned}$$

Hence $\nu_k(j, j') \in (\nu_{nk}(j, j'), \nu_{mk}(j, j')) = OI_k(j, j')$ if $|\lambda_{Y_1R}(i, 2)| < \infty$. Using similar arguments, we can show that $\nu_j(k, k') \in (\nu_{nj}(k, k'), \nu_{mj}(k, k')) = OI_j(k, k')$ if $|\lambda_{Y_1R}(i, 2)| < \infty$. Thus under Model C1, both the following conditions hold :

- (i) $\nu_k(j, j') \in OI_k(j, j')$ if $|\lambda_{Y_1R}(i, 2)| < \infty$,
- (ii) $\nu_j(k, k') \in OI_j(k, k')$ if $|\lambda_{Y_1R}(i, 2)| < \infty$.

2. Model C2 (MAR for Y_1) :

Under Model C2, we have

$$\begin{aligned}\nu_{ik}(j, j') &= \exp\{\lambda_{Y_2}(j) - \lambda_{Y_2}(j') + \lambda_{Y_1Y_2}(i, j) - \lambda_{Y_1Y_2}(i, j') + \lambda_{Y_2Y_3}(j, k) \\ &\quad - \lambda_{Y_2Y_3}(j', k) + \lambda_{Y_2R}(j, 1) - \lambda_{Y_2R}(j', 1)\}, \\ \nu_k(j, j') &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_2Y_3}(j, k) + \lambda_{Y_2R}(j, 2)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_2}(j') + \lambda_{Y_1Y_2}(i, j') + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_2Y_3}(j', k) + \lambda_{Y_2R}(j', 2)\}}, \\ \frac{\nu_{mk}(j, j')}{\nu_k(j, j')} &= \exp[2\{\lambda_{Y_2R}(j', 2) - \lambda_{Y_2R}(j, 2)\}] \times A_{mk}(j, j'), \\ \frac{\nu_{nk}(j, j')}{\nu_k(j, j')} &= \exp[2\{\lambda_{Y_2R}(j', 2) - \lambda_{Y_2R}(j, 2)\}] \times A_{nk}(j, j'),\end{aligned}$$

where

$$\begin{aligned}A_{mk}(j, j') &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j') + \lambda_{Y_1Y_2}(m, j) + \lambda_{Y_1Y_3}(i, k)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(m, j') + \lambda_{Y_1Y_3}(i, k)\}}, \\ A_{nk}(j, j') &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j') + \lambda_{Y_1Y_2}(n, j) + \lambda_{Y_1Y_3}(i, k)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(n, j') + \lambda_{Y_1Y_3}(i, k)\}}.\end{aligned}$$

Now $\nu_{ik}(j, j') < \nu_{mk}(j, j') \Rightarrow A_{mk}(j, j') > 1$ and $\nu_{ik}(j, j') > \nu_{nk}(j, j') \Rightarrow A_{nk}(j, j') < 1$. Suppose $\nu_k(j, j') \in OI_k(j, j') \Leftrightarrow \frac{\nu_{mk}(j, j')}{\nu_k(j, j')} > 1$ and $\frac{\nu_{nk}(j, j')}{\nu_k(j, j')} < 1$. Then we have

$$-\frac{1}{2} \log A_{mk}(j, j') < \lambda_{Y_2R}(j', 2) - \lambda_{Y_2R}(j, 2) < -\frac{1}{2} \log A_{nk}(j, j').$$

Next, consider the response and non-response odds based on π for any pair (k, k') of Y_3 and $1 \leq j \leq J$. Then

$$\begin{aligned}\nu_{ij}(k, k') &= \exp\{\lambda_{Y_3}(k) - \lambda_{Y_3}(k') + \lambda_{Y_1Y_3}(i, k) - \lambda_{Y_1Y_3}(i, k') + \lambda_{Y_2Y_3}(j, k) \\ &\quad - \lambda_{Y_2Y_3}(j, k')\}, \\ \nu_j(k, k') &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_3}(k) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_2Y_3}(j, k)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_3}(k') + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k') + \lambda_{Y_2Y_3}(j, k')\}}, \\ \frac{\nu_{mj}(k, k')}{\nu_j(k, k')} &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k') + \lambda_{Y_1Y_3}(m, k)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_1Y_3}(m, k')\}}, \\ \frac{\nu_{nj}(k, k')}{\nu_j(k, k')} &= \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k') + \lambda_{Y_1Y_3}(n, k)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_1Y_3}(n, k')\}}.\end{aligned}$$

Now it can be shown that $\nu_{ij}(k, k') < \nu_{mj}(k, k') \Rightarrow \nu_{mj}(k, k') > \nu_j(k, k')$ and $\nu_{ij}(k, k') > \nu_{nj}(k, k') \Rightarrow \nu_{nj}(k, k') < \nu_j(k, k')$. Hence $\nu_j(k, k') \in (\nu_{nj}(k, k'), \nu_{mj}(k, k')) = OI_j(k, k')$. So under Model C2, both the following conditions hold:

- (i) $\nu_k(j, j') \in OI_k(j, j')$ iff $-\frac{1}{2} \log A_{mk}(j, j') < \lambda_{Y_2R}(j', 2) - \lambda_{Y_2R}(j, 2) < -\frac{1}{2} \log A_{nk}(j, j')$,

(ii) $\nu_j(k, k') \in OI_j(k, k')$.

The conditions under which $\nu_k(j, j') \in OI_k(j, j')$ and $\nu_j(k, k') \in OI_j(k, k')$ can be obtained similarly for Models C3 and C4.

The following theorems provide necessary conditions for the missing mechanism of a variable in an $I \times J \times K \times 2$ table.

Theorem 3.1. Suppose Y_1 has a NMAR or MCAR mechanism in an $I \times J \times K \times 2$ table. Then we have the following cases:

- (i) $\nu_k(j, j') \in OI_k(j, j')$ if $|\lambda_{Y_1 R}(i, 1)| < \infty$,
- (ii) $\nu_j(k, k') \in OI_j(k, k')$ if $|\lambda_{Y_1 R}(i, 1)| < \infty$.

Proof. The proof follows from the discussions on Models C1 and C4. □

Henceforth, Condition (L1, L2) holds means both conditions L1 and L2 hold.

Theorem 3.2. Suppose Y_1 has a MAR mechanism in an $I \times J \times K \times 2$ table. Then only one of the Conditions (1a,1b) and (2a,2b) holds:

- 1a. For each k and each pair (j, j') of Y_2 , only one of the conditions below holds:
 - (i) $\nu_k(j, j') \in OI_k(j, j')$ iff $-\frac{1}{2} \log A_{mk}(j, j') < \lambda_{Y_2 R}(j', 2) - \lambda_{Y_2 R}(j, 2) < -\frac{1}{2} \log A_{nk}(j, j')$,
 - (ii) $\nu_k(j, j') \notin OI_k(j, j')$ iff $\lambda_{Y_2 R}(j', 2) - \lambda_{Y_2 R}(j, 2) > -\frac{1}{2} \log A_{nk}(j, j')$ or $\lambda_{Y_2 R}(j', 2) - \lambda_{Y_2 R}(j, 2) < -\frac{1}{2} \log A_{mk}(j, j')$,
- 1b. $\nu_j(k, k') \in OI_j(k, k')$;
- 2a. $\nu_k(j, j') \in OI_k(j, j')$,
- 2b. For each j and each pair (k, k') of Y_3 , only one of the conditions below holds:
 - (i) $\nu_j(k, k') \in OI_j(k, k')$ iff $-\frac{1}{2} \log A_{mj}(k, k') < \lambda_{Y_3 R}(k', 2) - \lambda_{Y_3 R}(k, 2) < -\frac{1}{2} \log A_{nj}(k, k')$,
 - (ii) $\nu_j(k, k') \notin OI_j(k, k')$ iff $\lambda_{Y_3 R}(k', 2) - \lambda_{Y_3 R}(k, 2) > -\frac{1}{2} \log A_{nj}(k, k')$ or $\lambda_{Y_3 R}(k', 2) - \lambda_{Y_3 R}(k, 2) < -\frac{1}{2} \log A_{mj}(k, k')$.

Proof. The proof follows from the discussions on Models C2 and C3. □

Remark 3.1. Note that if $\lambda_{Y_2 R}(j, r) = 0 \forall j, r$, then the MAR missing mechanism of Y_1 reduces to the MCAR mechanism. Also, $\nu_{mk}(j, j')/\nu_k(j, j') = A_{mk}(j, j') > 1$ and $\nu_{nk}(j, j')/\nu_k(j, j') = A_{nk}(j, j') < 1$ imply $\nu_k(j, j') \in OI_k(j, j')$, which is exactly one of the conditions under MCAR model for Y_1 (Model C4). Further note that $A_{mk}(j, j')$ and $A_{nk}(j, j')$ are independent of $\lambda_{Y_2}(j)$'s and $\lambda_R(r)$'s. Similar results hold when $\lambda_{Y_3 R}(k, r) = 0 \forall k, r$.

Remark 3.2. From Theorem 3.1, note that if the missing mechanism of Y_1 is NMAR or MCAR, then $\nu_k(j, j') \in OI_k(j, j')$ for any pair (j, j') of Y_2 and $\nu_j(k, k') \in OI_j(k, k')$ for any pair (k, k') of Y_3 . Also, if there exists at least one k or at least one pair (j, j') of Y_2 such that $\nu_k(j, j') \notin OI_k(j, j')$, then $|\lambda_{Y_2 R}(j', 2) - \lambda_{Y_2 R}(j, 2)|$ is larger than that when $\nu_k(j, j') \in OI_k(j, j')$. We say that the missing mechanism of Y_1 is strong MAR in the first case and non-strong (weak) in the second one. Similar results follow when we consider $\lambda_{Y_3 R}(k, 2)$'s.

Remark 3.3. If $I = J = K = 2$, then we have a $2 \times 2 \times 2 \times 2$ table. Under Models C2 and C3, we have

$$\lambda_{Y_1 Y_2}(1, 1) = \frac{1}{4} \log \left[\frac{\nu_{1k}(1, 2)}{\nu_{2k}(1, 2)} \right], \quad \lambda_{Y_1 Y_3}(1, 1) = \frac{1}{4} \log \left[\frac{\nu_{1j}(1, 2)}{\nu_{2j}(1, 2)} \right].$$

Hence, for fixed π , the length of $OI_k(1, 2) = |\nu_{1k}(1, 2) - \nu_{2k}(1, 2)|$ or that of $OI_j(1, 2) = |\nu_{1j}(1, 2) - \nu_{2j}(1, 2)|$, that is, the size of the parameter region for the weak MAR mechanism of Y_1 is directly proportional to $|\lambda_{Y_1 Y_2}(1, 1)|$ or $|\lambda_{Y_1 Y_3}(1, 1)|$, the strength of the associations between Y_1 and Y_2 or Y_3 respectively.

3.1.2. Assessment of the MCAR, NMAR and MAR mechanisms. It can be shown that perfect fits for fully and partially observed counts occur under Model C1 and not under Models C2, C3 and C4 (see Ghosh and Vellaisamy (2016)). This implies the MLE's are $\hat{\pi}_{ijk1} = y_{ijk1}/N$ and $\hat{\pi}_{+jk2} = y_{+jk2}/N$. Hence, the estimators of the various response and non-response odds under Model C1 are as follows.

$$\begin{aligned}\hat{\nu}_{ik}(j, j') &= \frac{y_{ijk1}}{y_{ij'k1}}, \hat{\nu}_{nk}(j, j') = \min_i \{\hat{\nu}_{ik}(j, j')\}, \hat{\nu}_{mk}(j, j') = \max_i \{\hat{\nu}_{ik}(j, j')\}, \hat{\nu}_k(j, j') = \frac{y_{+jk2}}{y_{+j'k2}}; \\ \hat{\nu}_{ij}(k, k') &= \frac{y_{ijk1}}{y_{ijk'1}}, \hat{\nu}_{nj}(k, k') = \min_i \{\hat{\nu}_{ij}(k, k')\}, \hat{\nu}_{mj}(k, k') = \max_i \{\hat{\nu}_{ij}(k, k')\}, \hat{\nu}_j(k, k') = \frac{y_{+jk2}}{y_{+jk'2}}\end{aligned}$$

The MLE's of the response and the nonresponse odds under non-perfect fit models are more involved than those under perfect fit models. For example, the estimated cell probabilities under Model C4 (see Ghosh and Vellaisamy (2016)) are

$$\hat{\pi}_{ijk1} = \frac{y_{ijk1}y_{+jk}y_{+++1}}{Ny_{+jk1}y_{++++}}, \quad \hat{\pi}_{+jk2} = \frac{y_{+++2} \sum_i \hat{\pi}_{ijk1}}{y_{+++1}} = \frac{y_{+++2}y_{+jk+}}{Ny_{++++}}.$$

Hence, the MLE's of the above odds under Model C4 are

$$\begin{aligned}\hat{\nu}_{ik}(j, j') &= \frac{y_{ijk1}y_{+jk}y_{+j'k1}}{y_{ij'k1}y_{+j'k+}y_{+jk1}}, \hat{\nu}_{nk}(j, j') = \min_i \{\hat{\nu}_{ik}(j, j')\}, \hat{\nu}_{mk}(j, j') = \max_i \{\hat{\nu}_{ik}(j, j')\}, \\ \hat{\nu}_k(j, j') &= \frac{\sum_i \hat{\pi}_{ijk1}}{\sum_i \hat{\pi}_{ij'k1}} = \frac{y_{+jk+}}{y_{+j'k+}}; \\ \hat{\nu}_{ij}(k, k') &= \frac{y_{ijk1}y_{+jk}y_{+jk'1}}{y_{ijk'1}y_{+jk'1}y_{+jk1}}, \hat{\nu}_{nj}(k, k') = \min_i \{\hat{\nu}_{ij}(k, k')\}, \hat{\nu}_{mj}(k, k') = \max_i \{\hat{\nu}_{ij}(k, k')\}, \\ \hat{\nu}_j(k, k') &= \frac{\sum_i \hat{\pi}_{ijk1}}{\sum_i \hat{\pi}_{ijk'1}} = \frac{y_{+jk+}}{y_{+jk'+}}.\end{aligned}$$

Denote the estimators of $OI_k(j, j')$ and $OI_j(k, k')$ by $\widehat{OI}_k(j, j') = (\hat{\nu}_{nk}(j, j'), \hat{\nu}_{mk}(j, j'))$ and $\widehat{OI}_j(k, k') = (\hat{\nu}_{nj}(k, k'), \hat{\nu}_{mj}(k, k'))$ respectively. Then the corollary below follows from Theorem 3.1 and Remark 3.2.

Corollary 3.1. For an $I \times J \times K \times 2$ table, if there exists at least one k or at least one pair (j, j') of Y_2 such that $\hat{\nu}_k(j, j') \notin \widehat{OI}_k(j, j')$, or there exists at least one j or at least one pair (k, k') of Y_3 such that $\hat{\nu}_j(k, k') \notin \widehat{OI}_j(k, k')$, then the missing data mechanism of Y_1 is more likely to be MAR, but neither NMAR nor MCAR.

3.2. Case 2: Two of the variables are missing. WLOG, suppose Y_1 and Y_2 are missing and for $i = 1, 2$, let R_i denote the missing indicator for Y_i such that $R_i = 1$ if Y_i is observed and $R_i = 2$ otherwise. Then for Y_1, Y_2, Y_3, R_1 and R_2 , we have an $I \times J \times K \times 2 \times 2$ table with cell counts $\mathbf{y} = \{y_{ijkrs}\}$ where $1 \leq i \leq I, 1 \leq j \leq J, 1 \leq k \leq K$ and $r, s = 1, 2$. The vector of observed counts is $\mathbf{y}_{\text{obs}} = (\{y_{ijk11}\}, \{y_{+jk21}\}, \{y_{i+k12}\}, \{y_{++k22}\})$. For $I = J = K = 2$, the $2 \times 2 \times 2 \times 2 \times 2$ incomplete table is given below.

Table 4. $2 \times 2 \times 2 \times 2 \times 2$ Incomplete Table.

				$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$	$Y_2 = 1$	y_{11111}	y_{11211}
			$Y_2 = 2$	y_{12111}	y_{12211}
		$R_2 = 2$	Missing	y_{1+112}	y_{1+212}
	$Y_1 = 2$	$R_2 = 1$	$Y_2 = 1$	y_{21111}	y_{21211}
			$Y_2 = 2$	y_{22111}	y_{22211}
		$R_2 = 2$	Missing	y_{2+112}	y_{2+212}
$R_1 = 2$	Missing	$R_2 = 1$	$Y_2 = 1$	y_{+1121}	y_{+1221}
			$Y_2 = 2$	y_{+2121}	y_{+2221}
		$R_2 = 2$	Missing	y_{++122}	y_{++222}

Let $\pi = \{\pi_{ijkrs}\}$ be the vector of cell probabilities, $\mu = \{\mu_{ijkrs}\}$ be the vector of expected counts and N be the total cell count. Under Poisson sampling, the log-likelihood kernel of μ is

$$\begin{aligned}
l(\mu; \mathbf{y}_{\text{obs}}) &= \sum_{i,j,k} y_{ijk11} \log \mu_{ijk11} + \sum_{j,k} y_{+jk21} \log \mu_{+jk21} + \sum_{i,k} y_{i+k12} \log \mu_{i+k12} \\
&\quad + \sum_k y_{++k22} \log \pi_{++k22} - \sum_{i,j,k,r,s} \mu_{ijkrs}.
\end{aligned}$$

The log-linear model (with no three-way or higher order interactions) in this case is

$$\begin{aligned}
\log \mu_{ijkrs} &= \lambda + \lambda_{Y_1}(i) + \lambda_{Y_2}(j) + \lambda_{Y_3}(k) + \lambda_{R_1}(x) + \lambda_{R_2}(s) + \lambda_{Y_1 Y_2}(i, j) \\
&\quad + \lambda_{Y_1 Y_3}(i, k) + \lambda_{Y_2 Y_3}(j, k) + \lambda_{Y_1 R_1}(i, r) + \lambda_{Y_2 R_1}(j, r) + \lambda_{Y_3 R_1}(k, r) \\
(3.3) \quad &\quad + \lambda_{Y_1 R_2}(i, s) + \lambda_{Y_2 R_2}(j, s) + \lambda_{Y_3 R_2}(k, s) + \lambda_{R_1 R_2}(r, s).
\end{aligned}$$

Three-way and higher order associations are assumed to be zero in (3.3) as they are difficult to interpret and analysis by ML estimation (without using iterative procedures) becomes intractable. Note that association terms among Y_i 's and those among R_i 's do not play a role in studying the missing data mechanisms of Y_i 's in (3.3). Also, the missingness mechanism of a variable cannot both be NMAR and MAR simultaneously. So log-linear parameters of the form $\lambda_{Y_i R_i R_j}$ for $i \neq j$ are excluded in (3.3). According to the MAR assumption in Case 1, the missing mechanism of a variable depends on only one of the other variables. So log-linear parameters of the form $\lambda_{Y_i Y_j R_i}$ for $i \neq j$ are also excluded in (3.3). Each log-linear parameter in (3.3) satisfies the constraint that the sum over each of its arguments is 0. Based on the assumption regarding various missing mechanisms of a variable in Case 1, there are 16 identifiable missing data models, which are submodels of (3.3) and categorized as follows.

- D1. MCAR model for both Y_1 and Y_2 (1 model),
- D2. NMAR model for both Y_1 and Y_2 (1 model),
- D3. MAR models for both Y_1 and Y_2 (4 models),
- D4. Mixture of MCAR and NMAR models for Y_1 and Y_2 (2 models),
- D5. Mixture of MCAR and MAR models for Y_1 and Y_2 (4 models),
- D6. Mixture of NMAR and MAR models for Y_1 and Y_2 (4 models).

3.2.1. *Properties of the missing data models.* Define the odds $\nu'_{ij}(k, k')$, $\nu'_{mj}(k, k')$, $\nu'_{nj}(k, k')$, $\nu'_j(k, k')$, $\nu'_{ik}(j, j')$, $\nu'_{mk}(j, j')$, $\nu'_{nk}(j, j')$ and $\nu'_k(j, j')$ similarly as the corresponding ones defined for the case when Y_1 is missing in an $I \times J \times K \times 2$ table. In this case, replace π_{ijk1} by π_{ijk11} , π_{+jk2} by π_{+jk21} , $\pi_{ij'k1}$ by $\pi_{ij'k11}$, $\pi_{+j'k2}$ by $\pi_{+j'k21}$, $\pi_{ijk'1}$ by $\pi_{ijk'11}$ and $\pi_{+jk'2}$ by $\pi_{+jk'21}$. Also define the following response and non-response odds based on π .

$$\omega'_{jk}(i, i') = \frac{\pi_{ijk11}}{\pi_{i'jk11}}, \omega'_{nk}(i, i') = \min_j \{\omega'_{jk}(i, i')\}, \omega'_{mk}(i, i') = \max_j \{\omega'_{jk}(i, i')\}, \omega'_k(i, i') = \frac{\pi_{i+k12}}{\pi_{i'+k12}};$$

$$\omega'_{ji}(k, k') = \frac{\pi_{ijk11}}{\pi_{ijk'11}}, \omega'_{ni}(k, k') = \min_j \{\omega'_{ji}(k, k')\}, \omega'_{mi}(k, k') = \max_j \{\omega'_{ji}(k, k')\}, \omega'_i(k, k') = \frac{\pi_{i+k12}}{\pi_{i'+k12}}$$

Let $OI'_k(i, i') = (\omega'_{nk}(i, i'), \omega'_{mk}(i, i'))$, $OI'_i(k, k') = (\omega'_{ni}(k, k'), \omega'_{mi}(k, k'))$, $OI'_k(j, j') = (\nu'_{nk}(j, j'), \nu'_{mk}(j, j'))$ and $OI'_j(k, k') = (\nu'_{nj}(k, k'), \nu'_{mj}(k, k'))$. Applying the methods described for $I \times J \times 2 \times 2$ and $I \times J \times K \times 2$ tables, the conditions under which the non-response odds belong to the open intervals formed by the response odds for Models D1-D6 may be obtained. Define

$$A'_{mk}(j, j') = \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j') + \lambda_{Y_1Y_2}(m, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_1R_2}(i, 1)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(m, j') + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_1R_2}(i, 1)\}} > 1,$$

$$A'_{mj}(k, k') = \frac{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k') + \lambda_{Y_1Y_3}(m, k) + \lambda_{Y_1R_2}(i, 1)\}}{\sum_i \exp\{\lambda_{Y_1}(i) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_3}(i, k) + \lambda_{Y_1Y_3}(m, k') + \lambda_{Y_1R_2}(i, 1)\}} > 1,$$

$$B'_{mi}(k, k') = \frac{\sum_i \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_2Y_3}(j, k') + \lambda_{Y_2Y_3}(m, k) + \lambda_{Y_2R_1}(j, 1)\}}{\sum_i \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_2Y_3}(j, k) + \lambda_{Y_2Y_3}(m, k') + \lambda_{Y_2R_1}(j, 1)\}} > 1,$$

$$B'_{mk}(i, i') = \frac{\sum_i \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i', j) + \lambda_{Y_1Y_2}(i, m) + \lambda_{Y_2Y_3}(j, k) + \lambda_{Y_2R_1}(j, 1)\}}{\sum_i \exp\{\lambda_{Y_2}(j) + \lambda_{Y_1Y_2}(i, j) + \lambda_{Y_1Y_2}(i', m) + \lambda_{Y_2Y_3}(j, k) + \lambda_{Y_2R_1}(j, 1)\}} > 1.$$

Similarly, define $A'_{nk}(j, j')$, $A'_{nj}(k, k')$, $B'_{ni}(k, k')$ and $B'_{nk}(i, i')$, each less than 1, by replacing m by n in the above expressions. For some of the above models, it can be shown that $\lambda_{Y_2R_1}(j, 1) = 0$ in $B'_{mi}(k, k')$, $B'_{mk}(i, i')$, $B'_{ni}(k, k')$ and $B'_{nk}(i, i')$ or $\lambda_{Y_1R_2}(i, 1) = 0$ in $A'_{mk}(j, j')$, $A'_{mj}(k, k')$, $A'_{nk}(j, j')$ and $A'_{nj}(k, k')$. Denote A^* to be A^+ (A' with $\lambda_{Y_1R_2}(j, 1) \neq 0$) or A^- (A' with $\lambda_{Y_1R_2}(i, 1) = 0$). Also, denote B^* to be B^+ (B' with $\lambda_{Y_2R_1}(j, 1) \neq 0$) or B^- (B' with $\lambda_{Y_2R_1}(j, 1) = 0$). Then the next result provides necessary conditions for the missing mechanism of a variable in an $I \times J \times K \times 2 \times 2$ table.

Theorem 3.3. Under the missing data models D1-D6 for an $I \times J \times K \times 2 \times 2$ table, we have the following cases corresponding to the missing mechanism of Y_1 .

(a) If Y_1 has a NMAR or MCAR mechanism, then

- (i) $\nu'_k(j, j') \in OI'_k(j, j')$ if $|\lambda_{Y_1R_1}(i, 2)| < \infty$,
- (ii) $\nu'_j(k, k') \in OI'_k(k, k')$ if $|\lambda_{Y_1R_1}(i, 2)| < \infty$.

(b) If Y_1 has a MAR mechanism, then only one of the Conditions (1a,1b) and (2a,2b) holds:

1a. For each k and each pair (j, j') of Y_2 , only one of the conditions below holds:

- (i) $\nu'_k(j, j') \in OI'_k(j, j')$ iff $-\frac{1}{2} \log A^*_{mk}(j, j') < \lambda_{Y_2R_1}(j', 2) - \lambda_{Y_2R_1}(j, 2) < -\frac{1}{2} \log A^*_{nk}(j, j')$,
- (ii) $\nu'_k(j, j') \notin OI'_k(j, j')$ iff $\lambda_{Y_2R_1}(j', 2) - \lambda_{Y_2R_1}(j, 2) > -\frac{1}{2} \log A^*_{nk}(j, j')$ or $\lambda_{Y_2R_1}(j', 2) - \lambda_{Y_2R_1}(j, 2) < -\frac{1}{2} \log A^*_{mk}(j, j')$,

1b. $\nu'_j(k, k') \in OI'_j(k, k')$;

2a. $\nu'_k(j, j') \in OI'_k(j, j')$,

2b. For each j and each pair (k, k') of Y_3 , only one of the conditions below holds:

- (i) $\nu'_j(k, k') \in OI'_j(k, k')$ iff $-\frac{1}{2} \log A^*(k, k') < \lambda_{Y_3 R_1}(k', 2) - \lambda_{Y_3 R_1}(j, 2) < -\frac{1}{2} \log A_{nj}^*(k, k')$,
- (ii) $\nu'_j(k, k') \notin OI'_j(k, k')$ iff $\lambda_{Y_3 R_1}(k', 2) - \lambda_{Y_3 R_1}(k, 2) > -\frac{1}{2} \log A_{nj}^*(k, k')$ or $\lambda_{Y_3 R_1}(k', 2) - \lambda_{Y_3 R_1}(k, 2) < -\frac{1}{2} \log A_{mj}^*(k, k')$.

A similar result for the various missing mechanisms of Y_2 under Models D1-D6 in an $I \times J \times K \times 2 \times 2$ table can be obtained. Also, some remarks regarding the relationship between MAR and MCAR mechanisms, degree of MAR mechanism and dependence of the size of the parameter region for the weak MAR mechanism on the strength of association between variables (in a $2 \times 2 \times 2 \times 2 \times 2$ table) can be made on the lines of Remarks 3.1-3.3.

3.2.2. Assessment of the MCAR, NMAR and MAR mechanisms. Here, we propose a method to assess the MCAR, NMAR and MAR assumptions in an $I \times J \times K \times 2 \times 2$ table. It can be shown that perfect fits for fully and partially observed data occur for models in Categories D2 and D6 (see Ghosh and Vellaisamy (2016)). So, the MLE's of π_{ijk11} , π_{i+k12} and π_{+jk21} under the above models are y_{ijk11}/N , y_{i+k12}/N and y_{+jk21}/N respectively. Hence, the estimators $\hat{\nu}'_{ij}(k, k')$, $\hat{\nu}'_{mj}(k, k')$, $\hat{\nu}'_{nj}(k, k')$, $\hat{\nu}'_j(k, k')$, $\hat{\nu}'_{ik}(j, j')$, $\hat{\nu}'_{mk}(j, j')$, $\hat{\nu}'_{nk}(j, j')$ and $\hat{\nu}'_k(j, j')$ are similar to the corresponding ones defined for the case when Y_1 is missing in an $I \times J \times K \times 2$ table. In this case, replace y_{ijk1} by y_{ijk11} , y_{+jk2} by y_{+jk21} , $y_{ij'k1}$ by $y_{ij'k11}$, $y_{+j'k2}$ by $y_{+j'k21}$, $y_{ijk'1}$ by $y_{ijk'11}$ and $y_{+jk'2}$ by $y_{+jk'21}$. Also, the estimators of the odds ω 's defined in the previous subsection are given below.

$$\begin{aligned}\hat{\omega}'_{jk}(i, i') &= \frac{y_{ijk11}}{y_{i'jk11}}, \hat{\omega}'_{nk}(i, i') = \min_j \{\hat{\omega}'_{jk}(i, i')\}, \hat{\omega}'_{mk}(i, i') = \max_j \{\hat{\omega}'_{jk}(i, i')\}, \hat{\omega}'_k(i, i') = \frac{y_{i+k12}}{y_{i'+k12}}; \\ \hat{\omega}'_{ji}(k, k') &= \frac{y_{ijk11}}{y_{ijk'11}}, \hat{\omega}'_{ni}(k, k') = \min_j \{\hat{\omega}'_{ji}(k, k')\}, \hat{\omega}'_{mi}(k, k') = \max_j \{\hat{\omega}'_{ji}(k, k')\}, \hat{\omega}'_i(k, k') = \frac{y_{i+k12}}{y_{i'+k12}}\end{aligned}$$

The estimated expected counts and hence the response and the nonresponse odds under non-perfect fit models are more complicated than those under perfect-fit models. For example, the MLE's of the cell probabilities under a model in D4 (MCAR for Y_1 , NMAR for Y_2) are (see Ghosh and Vellaisamy (2016))

$$\hat{\pi}_{ijk11} = \frac{y_{ijk11}y_{+++11}y_{+jk+1}}{Ny_{++++1}y_{+jk11}}, \quad \hat{\pi}_{+jk21} = \frac{y_{+++21} \sum_i \hat{\pi}_{ijk11}}{y_{++++1}} = \frac{y_{+jk+1}y_{+++21}}{Ny_{++++1}}, \quad \hat{\pi}_{i+k12} = \frac{y_{i+k12}}{N}.$$

So the MLE's of the odds under the above model are

$$\begin{aligned}\hat{\nu}'_{ik}(j, j') &= \frac{y_{ijk11}y_{+jk+1}y_{+j'k11}}{y_{ij'k11}y_{+j'k+1}y_{+jk11}}, \hat{\nu}'_{nk}(j, j') = \min_i \{\hat{\nu}'_{ik}(j, j')\}, \hat{\nu}'_{mk}(j, j') = \max_i \{\hat{\nu}'_{ik}(j, j')\}, \\ \hat{\nu}'_k(j, j') &= \frac{y_{+jk+1}}{y_{+j'k+1}}; \\ \hat{\nu}'_{ij}(k, k') &= \frac{y_{ijk11}y_{+jk+1}y_{+jk'11}}{Ny_{ijk'11}y_{+jk'+1}y_{+jk11}}, \hat{\nu}'_{nj}(k, k') = \min_i \{\hat{\nu}'_{ij}(k, k')\}, \hat{\nu}'_{mj}(k, k') = \max_i \{\hat{\nu}'_{ij}(k, k')\}, \\ \hat{\nu}'_j(k, k') &= \frac{y_{+jk+1}}{y_{+jk'+1}}; \\ \hat{\omega}'_{jk}(i, i') &= \frac{y_{ijk11}}{y_{i'jk11}}, \hat{\omega}'_{nk}(i, i') = \min_j \{\hat{\omega}'_{jk}(i, i')\}, \hat{\omega}'_{mk}(i, i') = \max_j \{\hat{\omega}'_{jk}(i, i')\}, \hat{\omega}'_k(i, i') = \frac{y_{i+k12}}{y_{i'+k12}}; \\ \hat{\omega}'_{ji}(k, k') &= \frac{y_{ijk11}y_{+jk+1}y_{+jk'11}}{Ny_{ijk'11}y_{+jk'+1}y_{+jk11}}, \hat{\omega}'_{ni}(k, k') = \min_j \{\hat{\omega}'_{ji}(k, k')\}, \hat{\omega}'_{mi}(k, k') = \max_j \{\hat{\omega}'_{ji}(k, k')\},\end{aligned}$$

$$\widehat{\omega}'_i(k, k') = \frac{y_{i+k'12}}{y_{i+k'12}}.$$

The MLE's of the response and nonresponse odds under the other D4 model (NMAR for Y_1 , MCAR for Y_2) can similarly be obtained using the estimated expected counts (see Ghosh and Vellaisamy (2016)). Let $\widehat{OI}'_k(i, i') = (\widehat{\omega}'_{nk}(i, i'), \widehat{\omega}'_{mk}(i, i'))$, $\widehat{OI}'_i(k, k') = (\widehat{\omega}'_{ni}(k, k'), \widehat{\omega}'_{mi}(k, k'))$, $\widehat{OI}'_k(j, j') = (\widehat{\nu}'_{nk}(j, j'), \widehat{\nu}'_{mk}(j, j'))$ and $\widehat{OI}'_j(k, k') = (\widehat{\nu}'_{nj}(k, k'), \widehat{\nu}'_{mj}(k, k'))$. Then we have the following corollary.

Corollary 3.2. For an $I \times J \times K \times 2 \times 2$ table, if $\widehat{\nu}'_k(j, j') \notin \widehat{OI}'_k(j, j')$ or $\widehat{\nu}'_k(j, j') \notin \widehat{OI}'_k(j, j')$ or $\widehat{\omega}'_k(i, i') \notin \widehat{OI}'_k(i, i')$ or $\widehat{\omega}'_i(k, k') \notin \widehat{OI}'_i(k, k')$ for at least one of i, j, k or one of the pairs $(i, i'), (j, j'), (k, k')$, then the missing mechanism of Y_1 or Y_2 is more likely to be MAR, but neither NMAR nor MCAR.

3.3. Case 3: All three variables are missing. We omit the details for this case. Similar to the tables discussed earlier, we can characterize the missing mechanisms of a variable for an $I \times J \times K \times 2 \times 2 \times 2$ table. Also, a method to assess the MCAR, MAR and NMAR assumptions may be obtained as in Cases 1 and 2 using estimators of the response and non-response odds, and open intervals, which depend only on the observed cell counts.

4. DATA ANALYSIS

In this section, we analyze some real-life datasets to demonstrate our results in Sections 2 and 3.

Example 4.1. Consider the data in Table 5 below discussed in Baker *et al.* (1992), which cross-classifies mother's self-reported smoking status (Y_1) ($Y_1 = 1(2)$ for smoker (non-smoker)) with newborn's weight (Y_2) ($Y_2 = 1(2)$ if weight < 2500 grams (≥ 2500 grams)). The supplementary margins contain data on only smoking status, data on only newborn's weight and missing data on both variables.

Table 5. Birth weight and smoking : observed counts.

		$R_2 = 1$		$R_2 = 2$
		$Y_2 = 1$	$Y_2 = 2$	Y_2 missing
$R_1 = 1$	$Y_1 = 1$	4512	21009	1049
	$Y_1 = 2$	3394	24132	1135
$R_1 = 2$	Y_1 missing	142	464	1224

From Table 5, note that $\frac{142}{464} \notin (\frac{3394}{24132}, \frac{4512}{21009})$, while $\frac{1049}{1135} \in (\frac{21009}{24132}, \frac{4512}{3394})$. Hence, from Corollary 2.1, the missing data mechanism of Y_1 or Y_2 is more likely to be MAR, but neither NMAR nor MCAR. This result coincides with the analysis by Baker *et al.* (1992) who infer that the most parsimonious fit model is MAR for Y_1 and MCAR for Y_2 (Model M4 in Section 2). They also mention that boundary solutions occur on fitting NMAR models for Y_1 (Models M1, M2 and M3 in Section 2) to the dataset in Table 5. This implies that these models provide poor fits to the observed data (see Clarke and Smith (2005)), which further supports our observation.

Example 4.2. Consider Table 6 below discussed in Park *et al.* (2014), which cross-classifies data on bone mineral density (Y_1) and family income (Y_2) in a $3 \times 3 \times 2 \times 2$ incomplete table. Both variables Y_1 and Y_2 have three levels. The total count is 2998 out of which data on Y_1

and Y_2 are available for 1844 persons, data on Y_1 only for 231 persons, data on Y_2 only for 878 persons, and data on neither of them for 45 persons.

Table 6. Bone mineral density (Y_1) and family income (Y_2).

		$R_2 = 1$			$R_2 = 2$
		$Y_2 = 1$	$Y_2 = 2$	$Y_2 = 3$	Missing
$R_1 = 1$	$Y_1 = 1$	621	290	284	135
	$Y_1 = 2$	260	131	117	69
	$Y_1 = 3$	93	30	18	27
$R_1 = 2$	Missing	456	156	266	45

From Table 6, we have $\frac{456}{156} \in (\frac{260}{131}, \frac{93}{30})$, $\frac{456}{266} \notin (\frac{621}{284}, \frac{93}{18})$ and $\frac{156}{266} \notin (\frac{290}{284}, \frac{30}{18})$, while $\frac{135}{69} \notin (\frac{290}{131}, \frac{284}{117})$, $\frac{135}{27} \notin (\frac{621}{93}, \frac{284}{18})$ and $\frac{69}{27} \notin (\frac{260}{93}, \frac{117}{18})$. Hence, it follows from Corollary 2.1 that the missing mechanism of Y_1 or Y_2 is probably MAR, but neither NMAR nor MCAR. Let G^2 denote the likelihood ratio statistic for testing the goodness of fit of the proposed model against the model in (2.2). When we fit Models M1-M9 (see Section 2) to the data in Table 6 using ‘ecm.cat’ function of the ‘cat’ package in R software, we deduce that the best fit model is MAR for Y_1 and MCAR for Y_2 (Model M4), based on minimum $G^2 = 5.42$ and p -value = 0.07. This analysis supports our earlier observation. Park *et al.* (2014) also showed that boundary solutions occur on fitting NMAR models for Y_1 or Y_2 (Models M1, M2, M3, M6 and M8), which thereby provide poor fits to the given data and hence further evidence for our result.

Example 4.3. In this example, we consider a real-life example from Rubin *et al.* (1995). Based on the Slovenian public opinion (SPO) survey, the dataset shown in Table 7 below, is a $2 \times 2 \times 2 \times 2 \times 2 \times 2$ table classified by the variables Secession (Y_1), Attendance (Y_2) and Independence (Y_3), each having two levels Yes (1) and No (2). The “Don’t know” category (missing margins) for each variable is denoted by “Missing”. We replace the count 0 by 2 in the full table. The total cell count is 2076, consisting of data on all three variables observed ($R_1 = R_2 = R_3 = 1$) for 1456 persons, Y_1 and Y_2 observed ($R_1 = R_2 = 1, R_3 = 2$) for 57 persons, Y_1 and Y_3 observed ($R_1 = R_3 = 1, R_2 = 2$) for 171 persons, Y_2 and Y_3 observed ($R_2 = R_3 = 1, R_1 = 2$) for 95 persons, only Y_1 observed ($R_2 = R_3 = 2, R_1 = 1$) for 40 persons, only Y_2 observed ($R_1 = R_3 = 2, R_2 = 1$) for 134 persons, only Y_3 observed ($R_1 = R_2 = 2, R_3 = 1$) for 27 persons, and all missing ($R_1 = R_2 = R_3 = 2$) for 96 persons.

Table 7. Data from the SPO survey.

				$R_3 = 1$		$R_3 = 2$
				$Y_3 = 1$	$Y_3 = 2$	Missing
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$	$Y_2 = 1$	1191	8	21
			$Y_2 = 2$	8	2	4
		$R_2 = 2$	Missing	107	3	9
	$Y_1 = 2$	$R_2 = 1$	$Y_2 = 1$	158	68	29
			$Y_2 = 2$	7	14	3
		$R_2 = 2$	Missing	18	43	31
$R_1 = 2$	Missing	$R_2 = 1$	$Y_2 = 1$	90	2	109
			$Y_2 = 2$	1	2	25
		$R_2 = 2$	Missing	19	8	96

WLOG, consider the subtable of Table 7 in which data on only Y_1 is missing as given below.

Table 8. Subtable of Table 7 for Y_1 .

			$Y_3 = 1$	$Y_3 = 2$
$R = 1$	$Y_1 = 1$	$Y_2 = 1$	1191	8
		$Y_2 = 2$	8	2
	$Y_1 = 2$	$Y_2 = 1$	158	68
		$Y_2 = 2$	7	14
$R = 2$	Missing	$Y_2 = 1$	90	2
		$Y_2 = 2$	1	2

From Table 8, we observe that $\frac{90}{2} \in (\frac{158}{68}, \frac{1191}{8})$, $\frac{1}{2} \notin (\frac{7}{14}, \frac{8}{2})$, $\frac{90}{1} \in (\frac{158}{7}, \frac{1191}{8})$ and $\frac{2}{2} \notin (\frac{8}{2}, \frac{68}{14})$. Hence, from Corollary 3.1, the missing mechanism of Y_1 is probably MAR, but neither NMAR nor MCAR. Let G^2 denote the likelihood ratio statistic for testing the goodness of fit of the proposed model against the model in (3.2). On fitting Models C1-C4 (see Subsection 3.1) to the data in Table 8, based on minimum $G^2 = 0.96$ and p -value = 0.62, we infer that the best fit model is MAR for Y_1 (Model C3). This observation is consistent with our earlier result (Corollary 3.1).

Example 4.4. In this example, we use the data in Table 7 of Example 4.3. Consider WLOG the subtable of Table 7 in which data on Y_1 and Y_2 are missing as given below.

Table 9. Subtable of Table 7 for Y_1 and Y_2 .

			$Y_3 = 1$	$Y_3 = 2$
$R_1 = 1$	$Y_1 = 1$	$R_2 = 1$ $Y_2 = 1$	1191	8
		$Y_2 = 2$	8	2
		$R_2 = 2$ Missing	107	3
	$Y_1 = 2$	$R_2 = 1$ $Y_2 = 1$	158	68
		$Y_2 = 2$	7	14
		$R_2 = 2$ Missing	18	43
$R_1 = 2$	Missing	$R_2 = 1$ $Y_2 = 1$	90	2
		$Y_2 = 2$	1	2
		$R_2 = 2$ Missing	19	8

From Table 9, note that $\frac{90}{2} \in (\frac{158}{68}, \frac{1191}{8})$, $\frac{1}{2} \notin (\frac{7}{14}, \frac{8}{2})$, $\frac{90}{1} \in (\frac{158}{7}, \frac{1191}{8})$ and $\frac{2}{2} \notin (\frac{8}{2}, \frac{68}{14})$, while $\frac{107}{3} \in (\frac{8}{2}, \frac{1191}{8})$, $\frac{18}{43} \notin (\frac{7}{14}, \frac{158}{68})$, $\frac{107}{18} \in (\frac{8}{7}, \frac{1191}{158})$ and $\frac{3}{43} \notin (\frac{8}{68}, \frac{2}{14})$. Hence, from Corollary 3.2, we deduce that the missing mechanism of Y_1 or Y_2 is more likely to be MAR, but neither NMAR nor MCAR. Let G^2 denote the likelihood ratio statistic for testing the goodness of fit of the proposed model against the model in (3.3). When Models D1-D6 (see Subsection 3.2) are fitted to the data in Table 9, the best fit model is MAR for Y_1 (missing mechanism depends on Y_2) and NMAR for Y_2 based on minimum G^2 value = 2.18 and p -value = 0.7031. This validates our earlier observation from Corollary 3.2.

5. CONCLUSIONS

In this paper, the missing data models for $I \times J \times 2 \times 2$, $I \times J \times K \times 2$, $I \times J \times K \times 2 \times 2$ and $I \times J \times K \times 2 \times 2 \times 2$ tables are considered using hierarchical log-linear models. Some

particular properties of these missing data models are discussed in detail. We provide necessary conditions for the various missing mechanisms of a variable in terms of response and non-response odds. These conditions help us to establish simple and useful procedures based only on the observed counts in the tables, which aid in the evaluation of MCAR, MAR and NMAR mechanisms of the variables. Finally, some real-life data analysis illustrate our results. Note that the models, techniques and results in this paper can be extended to higher dimensional tables also.

We now provide some comments on our methods of assessment for the MCAR or NMAR or MAR mechanism of each missing variable in various incomplete tables. Our aim has been to use estimates of the response and nonresponse odds involving only the fully and partially observed counts respectively in the tables, which are easy to calculate and simplify the verification process. If the missing mechanism of at least one of the variables is MCAR, then calculation of estimates of the response and nonresponse odds becomes tedious and complicated. This is because such models do not provide perfect fits for the observed counts in the tables (see Ghosh and Vellaisamy (2016)). Also, for evaluation purposes, it is sufficient to consider models under which the missing mechanism of a variable is MCAR or NMAR.

Note that the methods proposed in this paper are a form of sensitivity analysis to assess the MCAR, NMAR and MAR assumptions of missing variables in an incomplete table. They are not intended to replace formal model selection for finding the best fit log-linear model for the given data. Such methods work well when only one of the variables is missing in an incomplete table (see Example 4.3). The best fit model is usually identified in this case. However, when two or more variables are missing in these tables, our methods can provide probable missing mechanism (MCAR or NMAR or MAR) for each of the variables, but not the exact best fit model (see Examples 4.1, 4.2 and 4.4).

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